

THE BALMER SPECTRUM OF A TAME STACK

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ABSTRACT. Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. We classify the thick \otimes -ideals of $D_{qc}(X)^c$. If X is tame, then we also compute the Balmer spectrum of the \otimes -triangulated category of perfect complexes on X . In addition, if X admits a coarse space X_{cs} , then we prove that the Balmer spectra of X and X_{cs} are naturally isomorphic.

1. INTRODUCTION

Let X be a quasi-compact and quasi-separated scheme. Let $\mathrm{Perf}(X)$ be the \otimes -triangulated category of perfect complexes on X . A celebrated result of Thomason [Tho97, Thm. 3.15], extending the work of Hopkins [Hop87, §4] and Neeman [Nee92a, Thm. 1.5], is a classification of the thick \otimes -ideals of $\mathrm{Perf}(X)$ in terms of the *Thomason* subsets of $|X|$, which are those subsets $Y \subseteq |X|$ expressible as a union $\cup_{\alpha} Y_{\alpha}$ such that $|X| \setminus Y_{\alpha}$ is quasi-compact and open.

If X is a quasi-compact and quasi-separated algebraic space, Deligne–Mumford stack, or algebraic stack, then it is also natural to consider the \otimes -triangulated category $\mathrm{Perf}(X)$ of perfect complexes on X (see [HR14c, §3] for precise definitions).

In general, Thomason’s classification of thick \otimes -ideals of $\mathrm{Perf}(X)$ fails for algebraic stacks (Example 3.2). If one instead works with the \otimes -ideal $D_{qc}(X)^c \subseteq \mathrm{Perf}(X)$ of *compact* perfect complexes, then the first main result of this article is that the classification goes through without change.

Theorem 1.1 (Classification of thick \otimes -ideals). *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Then there is a bijective correspondence between thick \otimes -ideals of $D_{qc}(X)^c$ and Thomason subsets of $|X|$.*

Some special cases of Theorem 1.1 are the following:

- if k is a field and G is a finite group, then $D^b(\mathrm{Proj} kG)$ has no non-trivial \otimes -ideals;
- if Y is a quasi-projective scheme over a field k with a proper action of a group scheme G , then the thick \otimes -ideals of $D(\mathrm{QCoh}^G(Y))^c$ are in bijective correspondence with the G -invariant Thomason subsets of X .

The first special case is easy to prove directly and is well-known (cf. [BIK11, Prop. 2.1]). In some sense, this makes our results orthogonal to [BIK11]. The second special case was only known in characteristic 0 when Y was normal or quasi-affine [Kri09, Thm. 7.8] or in characteristic p when G is finite of order prime to p and X is smooth [DM12, Thm. 1.2].

We prove Theorem 1.1 using tensor nilpotence with parameters (Theorem 2.3), which extends [Tho97, Thm. 3.8] and [Hop87, Thm. 10ii] (cf. [Nee92a, 1.1]) to quasi-compact algebraic stacks with quasi-finite and separated diagonal. As should be expected, stacks of the form $[Y/G]$, where Y is an affine variety over a field k and G is a finite group with order divisible by the characteristic of k , are the

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most troublesome. This is dealt with in Lemma 2.6, which relies on some results developed in Appendix A.

If \mathcal{T} is a \otimes -triangulated category, then Balmer [Bal05] has functorially constructed from \mathcal{T} a locally ringed space $\mathrm{Sp}^{\mathrm{Bal}}(\mathcal{T})$, the *Balmer spectrum*. A fundamental result of Balmer [Bal05, Thm. 5.5], which was extended by Buan–Krause–Solberg [BKS07, Thm. 9.5] to the non-noetherian setting, is that if X is a quasi-compact and quasi-separated scheme, then there is a naturally induced isomorphism

$$X \rightarrow \mathrm{Sp}^{\mathrm{Bal}}(\mathrm{Perf}(X)).$$

An algebraic stack is *tame* if its stabilizer groups at geometric points are finite linearly reductive group schemes [AOV08, Defn. 2.2]. Every scheme and algebraic space is tame. Moreover, in characteristic zero, a stack is Deligne–Mumford if and only if it is tame. In characteristic $p > 0$, there are non-tame Deligne–Mumford stacks (e.g., $B_{\mathbb{F}_p}(\mathbb{Z}/p\mathbb{Z})$) and tame stacks that are not Deligne–Mumford (e.g., $B_{\mathbb{F}_p}\mu_p$). Nagata’s Theorem [HR14b, Thm. 1.2] provides a classification of finite linearly reductive group schemes over fields, thus allows one to determine whether a given algebraic stack is tame. Note that our definition of tame stack is substantially weaker than that what appears in [AOV08, Defn. 3.1] (see Appendix A).

Tame stacks are precisely those stacks with quasi-finite diagonal such that the compact objects of $\mathrm{D}_{\mathrm{qc}}(X)$ coincide with the perfect complexes. Using Theorem 1.1, we extend the result of Buan–Krause–Solberg to tame stacks.

Theorem 1.2. *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. If X is tame, then there is a natural isomorphism of locally ringed spaces*

$$(|X|, \mathcal{O}_{X_{\mathrm{Zar}}}) \rightarrow \mathrm{Sp}^{\mathrm{Bal}}(\mathrm{Perf}(X)),$$

where $\mathcal{O}_{X_{\mathrm{Zar}}}$ is the Zariski sheaf $U \mapsto \Gamma(U, \mathcal{O}_X)$.

Theorem 1.2 implies that the Balmer spectrum cannot be used to reconstruct locally separated algebraic spaces [Knu71, Ex. 2]. Balmer [Bal13] has recently initiated the study of unramified monoids in \otimes -triangulated categories and Neeman [Nee14] has classified them in the case of a separated noetherian scheme. It is hoped that a refinement of the Balmer spectrum can be constructed from unramified monoids, which would—at least—permit the reconstruction of algebraic spaces.

If X is an algebraic stack with finite inertia (e.g. a separated Deligne–Mumford stack), then X admits a *coarse space* $\pi: X \rightarrow X_{\mathrm{cs}}$ [KM97, Ryd13], which is the universal map from X to an algebraic space. If X has finite inertia, then X has separated diagonal. Thus we can also establish the following.

Theorem 1.3. *Let X be a quasi-compact, quasi-separated algebraic stack with finite inertia and coarse space $\pi: X \rightarrow X_{\mathrm{cs}}$. If X is tame, then*

$$\mathrm{Sp}^{\mathrm{Bal}}(\mathrm{L}\pi^*): \mathrm{Sp}^{\mathrm{Bal}}(\mathrm{Perf}(X)) \rightarrow \mathrm{Sp}^{\mathrm{Bal}}(\mathrm{Perf}(X_{\mathrm{cs}}))$$

is an isomorphism of ringed spaces.

Krishna [Kri09, Thm. 7.10] proved Theorem 1.3 when X is of the form $[W/G]$, where W is quasi-projective and normal or quasi-affine, and G is a linear algebraic group in characteristic 0 acting properly on W . Dubey–Mallick [DM12, Thm. 1.2] proved a similar result in positive characteristic, but required W to be smooth and G a finite group with order not divisible by the characteristic of the ground field. In particular, Theorem 1.3 is stronger than all existing results and Theorems 1.1 and 1.2 are new.

Assumptions and conventions. A priori, we make no separation assumptions on our algebraic stacks. However, all stacks used in this article will be, at the least, quasi-compact and quasi-separated. Usually, they will also have separated diagonal. If X is an algebraic stack, then let $|X|$ denote its associated Zariski topological space [LMB, §5]. For derived categories of algebraic stacks, we use the conventions and notations of [HR14c, §1]. In particular, if X is an algebraic stack, then $\text{Mod}(X)$ is the abelian category of \mathcal{O}_X -modules on the lisse-étale site of X and $\text{D}_{\text{qc}}(X)$ denotes the unbounded derived category of $\mathcal{O}_{X_{\text{lisse-ét}}}$ -modules with quasi-coherent cohomology sheaves. If $f: X \rightarrow Y$ is a morphism of algebraic stacks, then there is always an adjoint pair of unbounded derived functors

$$\text{D}_{\text{qc}}(X) \begin{matrix} \xrightarrow{\text{R}(f_{\text{qc}})^*} \\ \xleftarrow{\text{L}f_{\text{qc}}^*} \end{matrix} \text{D}_{\text{qc}}(Y).$$

If f is quasi-compact, quasi-separated and representable, then $\text{R}(f_{\text{qc}})^*$ agrees with $\text{R}f_*$, the unbounded derived functor of $f_*: \text{Mod}(X) \rightarrow \text{Mod}(Y)$ [HR14c, Lem. 2.5(3) & Thm. 2.6(2)]. If f is flat, then $\text{L}f_{\text{qc}}^*$ agrees with the unique extension of the exact functor $f^*: \text{Mod}(Y) \rightarrow \text{Mod}(X)$ to the unbounded derived category.

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2. TENSOR NILPOTENCE WITH PARAMETERS

We begin with the following definition.

Definition 2.1. Let X be an algebraic stack and let $\xi: M \rightarrow N$ be a morphism in $\text{D}_{\text{qc}}(X)$. If $Z \subseteq |X|$ is a subset, then ξ *vanishes at the points of Z* if for every algebraically closed field k and morphism $z: \text{Spec } k \rightarrow X$ that factors through Z , then $\text{L}z_{\text{qc}}^* \xi$ is the zero map in $\text{D}_{\text{qc}}(\text{Spec } k)$.

The following lemma will connect this definition with a more familiar notion for schemes.

Lemma 2.2. *Let X be a scheme and let $\xi: M \rightarrow N$ be a morphism in $\text{D}_{\text{qc}}(X)$. If $Z \subseteq |X|$ is a subset, then ξ vanishes at the points of Z if and only if $\xi \otimes_{\mathcal{O}_X}^{\text{L}} \kappa(z)$ is the zero map in $\text{D}(\kappa(z))$ for every $z \in Z$, where $\kappa(z)$ denotes the residue field of z .*

Proof. We immediately reduce to the situation where $X = \text{Spec } \kappa$ and κ is a field. It now suffices to prove that if $\kappa \subseteq k$ is a field extension, where k is algebraically closed, then $\xi \otimes k$ is the zero map in $\text{D}(k)$ if and only if ξ is the zero map in $\text{D}(\kappa)$. This is obvious. \square

If $K \in \text{D}_{\text{qc}}(X)$, then the *cohomological support* of K is defined to be the subset

$$\text{supph}(K) = \bigcup_{n \in \mathbb{Z}} \text{supp}(\mathcal{H}^n(K)) \subseteq |X|.$$

For the basic properties of cohomological support, see [HR14c, Lem. 3.5], which extends [Tho97, Lem. 3.3] to algebraic stacks. The main result of this section is the following theorem.

Theorem 2.3 (Tensor nilpotence with parameters). *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Let $\phi: E \rightarrow F$ be a morphism in $\text{D}_{\text{qc}}(X)$, where $E \in \text{D}_{\text{qc}}(X)^c$. Let $K \in \text{Perf}(X)$. If ϕ vanishes at the points of $\text{supph}(K)$, then there exists a positive integer n such that $K \otimes_{\mathcal{O}_X}^{\text{L}} (\psi^{\otimes n}) = 0$ in $\text{D}_{\text{qc}}(X)$.*

The following example demonstrates that Theorem 2.3 cannot be weakened to the situation where $E \in \text{Perf}(X)$.

Example 2.4. Let $X = B_{\mathbb{F}_2}(\mathbb{Z}/2\mathbb{Z})$. That X is a quasi-compact, non-tame Deligne–Mumford stack with finite diagonal. Consider the adjunction morphism $\eta: \mathcal{O}_X \rightarrow x_*\mathcal{O}_{\mathbb{F}_2}$, where $x: \text{Spec } \mathbb{F}_2 \rightarrow X$ is the usual cover. Then there is a natural map $\phi: C \rightarrow \mathcal{O}_X[1]$, where $C = \text{coker}(\eta)$. Clearly, ϕ vanishes at the points of $|X|$ (because $x^*\eta$ is split). If $\phi^{\otimes n} = 0$ for some n , then it is easily determined that this implies that $\mathcal{O}_X \in D_{\text{qc}}(X)^c$, which is false.

Proof of Theorem 2.3. Let \mathbf{E} be the category of representable, quasi-finite, flat and separated morphisms of finite presentation over X . Let $\mathbf{D} \subseteq \mathbf{E}$ be the full subcategory with objects those $(U \xrightarrow{p} X)$ such that there exists an integer $n > 0$ with $p^*(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\psi^{\otimes n})) = 0$. It suffices to prove that $\mathbf{D} = \mathbf{E}$. By the induction principle (Theorem B.1), it is sufficient to verify the following three conditions:

- (I1) if $(U \rightarrow W) \in \mathbf{E}$ is an open immersion and $W \in \mathbf{D}$, then $U \in \mathbf{D}$;
- (I2) if $(V \rightarrow W) \in \mathbf{E}$ is finite and surjective, where V is an affine scheme, then $W \in \mathbf{D}$; and
- (I3) if $(U \xrightarrow{i} W)$, $(W' \xrightarrow{f} W) \in \mathbf{E}$, where i is an open immersion and f is étale and an isomorphism over $W \setminus U$, then $W \in \mathbf{D}$ whenever $U, W' \in \mathbf{D}$.

Now condition (I1) is trivial and condition (I3) is Lemma 2.5. For condition (I2), by Lemma 2.6, it remains to prove that every affine scheme belongs to \mathbf{D} . By Lemma 2.2 and [Tho97, Lem. 3.14] (or [Nee92a, Lem. 1.2]), the result follows. \square

Lemma 2.5. *Consider a 2-cartesian diagram of algebraic stacks*

$$\begin{array}{ccc} U' & \xrightarrow{i'} & W' \\ f_U \downarrow & & \downarrow f \\ U & \xrightarrow{i} & W, \end{array}$$

where W is quasi-compact and quasi-separated, i is a quasi-compact open immersion and f is representable, étale, finitely presented and an isomorphism over $X \setminus U$. Let $\psi: E \rightarrow F$ be a morphism in $D_{\text{qc}}(X)$ and let $K \in D_{\text{qc}}(X)$. For each integer $n > 0$, let $\phi_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\psi^{\otimes n})$. If $f^*\phi_n = 0$ and $i^*\phi_n = 0$, then $\phi_{2n} = 0$.

Proof. To simplify notation, we let $E_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} E^{\otimes n}$ and $F_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} F^{\otimes n}$. We will argue similarly to [Tho97, Thm. 3.6], but using the Mayer–Vietoris triangle for étale neighbourhoods of stacks developed in [HR14c, Lem. 5.6(1)] instead of [Tho97, Lem. 3.5].

Let $k = f \circ i'$. By [HR14c, Lem. 5.6(1)], there is a distinguished triangle in $D_{\text{qc}}(X)$:

$$F_n \longrightarrow Ri_*i^*F_n \oplus Rf_*f^*F_n \longrightarrow Rk_*k^*F_n \xrightarrow{d} F_n[1].$$

By applying the homological functor $\text{Hom}_{\mathcal{O}_X}(E_n, -)$ to the distinguished triangle above, we find that there exists a morphism $t: E_n \rightarrow Rk_*k^*F_n[-1]$ such that $\delta(t) =$

ϕ_n , where δ is the boundary map induced by d . But there is a commutative diagram

$$\begin{array}{ccc}
 & (\mathrm{R}k_*k^*F_n[-1]) \otimes_{\mathcal{O}_X}^{\mathrm{L}} E^{\otimes n} & \\
 t \otimes \mathrm{Id} \nearrow & \downarrow \mathrm{Id} \otimes \psi^{\otimes n} & \searrow \delta \otimes \psi^{\otimes n} \\
 E_n \otimes_{\mathcal{O}_X}^{\mathrm{L}} E^{\otimes n} & \xrightarrow{\phi_{2n}} & F_n \otimes_{\mathcal{O}_X}^{\mathrm{L}} F^{\otimes n} \\
 t \otimes \psi^{\otimes n} \searrow & & \nearrow \delta \otimes \mathrm{Id} \\
 & (\mathrm{R}k_*k^*F_n[-1]) \otimes_{\mathcal{O}_X}^{\mathrm{L}} F^{\otimes n}, &
 \end{array}$$

so it remains to prove that the vertical map above is zero. To see this, the projection formula [HR14c, Cor. 4.18] implies that we have a commutative diagram

$$\begin{array}{ccc}
 (\mathrm{R}k_*k^*F_n[-1]) \otimes_{\mathcal{O}_X}^{\mathrm{L}} E^{\otimes n} & \xrightarrow{\sim} & \mathrm{R}k_*k^*(K \otimes_{\mathcal{O}_X}^{\mathrm{L}} F^{\otimes n} \otimes_{\mathcal{O}_X}^{\mathrm{L}} E^{\otimes n}[-1]) \\
 \mathrm{Id} \otimes \psi^{\otimes n} \downarrow & & \downarrow \mathrm{R}k_*k^*(F^{\otimes n} \otimes \phi_n[-1]) \\
 (\mathrm{R}k_*k^*F_n[-1]) \otimes_{\mathcal{O}_X}^{\mathrm{L}} F^{\otimes n} & \xrightarrow{\sim} & \mathrm{R}k_*k^*(K \otimes_{\mathcal{O}_X}^{\mathrm{L}} F^{\otimes n} \otimes_{\mathcal{O}_X}^{\mathrm{L}} F^{\otimes n}[-1]).
 \end{array}$$

Since $k^*\phi_n = 0$, the result follows. \square

The following lemma is similar to a special case of [Ela11, Thm. 7.3 & Cor. 9.6]. Also, see [Kri09, Proof of Prop. 7.6] and [DM12, Lem. 3.8].

Lemma 2.6. *Let W be an algebraic stack and let $v: V \rightarrow W$ be a finite and faithfully flat morphism of finite presentation, where V is an affine scheme. Let $\phi: E \rightarrow F$ be a morphism in $\mathrm{D}_{\mathrm{qc}}(W)$, where $E \in \mathrm{D}_{\mathrm{qc}}(W)^c$. Let $K \in \mathrm{Perf}(W)$. If $v^*(K \otimes_{\mathcal{O}_W}^{\mathrm{L}} \psi) = 0$ in $\mathrm{D}_{\mathrm{qc}}(V)$, then $K \otimes_{\mathcal{O}_W}^{\mathrm{L}} \psi = 0$ in $\mathrm{D}_{\mathrm{qc}}(W)$.*

Proof. By [HR14c, Prop. 3.3], $\mathrm{R}(v_{\mathrm{qc}})_*$ admits a right adjoint v^\times and there is a functorial isomorphism $v^\times(\mathcal{O}_W) \otimes_{\mathcal{O}_V}^{\mathrm{L}} \mathrm{L}v_{\mathrm{qc}}^*(M) \simeq v^\times(M)$ for every $M \in \mathrm{D}_{\mathrm{qc}}(W)$. In particular, if $v^*(K \otimes_{\mathcal{O}_W}^{\mathrm{L}} \psi) = 0$ in $\mathrm{D}_{\mathrm{qc}}(V)$, then $v^\times(K \otimes_{\mathcal{O}_W}^{\mathrm{L}} \psi) = 0$ in $\mathrm{D}_{\mathrm{qc}}(V)$. By adjunction, it follows that the induced composition

$$\mathrm{R}(v_{\mathrm{qc}})_*v^\times(K \otimes_{\mathcal{O}_W}^{\mathrm{L}} E) \rightarrow K \otimes_{\mathcal{O}_W}^{\mathrm{L}} E \rightarrow K \otimes_{\mathcal{O}_W}^{\mathrm{L}} F$$

vanishes in $\mathrm{D}_{\mathrm{qc}}(W)$. Thus it suffices to prove that $\mathrm{R}(v_{\mathrm{qc}})_*v^\times(K \otimes_{\mathcal{O}_W}^{\mathrm{L}} E) \rightarrow K \otimes_{\mathcal{O}_W}^{\mathrm{L}} E$ admits a section. Since $E \in \mathrm{D}_{\mathrm{qc}}(W)^c$ and $K \in \mathrm{Perf}(W)$, it follows that $K \otimes_{\mathcal{O}_W}^{\mathrm{L}} E \in \mathrm{D}_{\mathrm{qc}}(W)^c$. Hence, we need only prove that if $M \in \mathrm{D}_{\mathrm{qc}}(W)^c$, then the trace morphism $\mathrm{Tr}_M: \mathrm{R}(v_{\mathrm{qc}})_*v^\times(M) \rightarrow M$ admits a section. By Lemma A.1, M is quasi-isomorphic to a direct summand of $\mathrm{R}(v_{\mathrm{qc}})_*P$, where $P \in \mathrm{Perf}(V)$. Thus we are reduced to proving that $\mathrm{Tr}_{\mathrm{R}(v_{\mathrm{qc}})_*P}$ admits a section. This is trivial and the result follows. \square

3. THE CLASSIFICATION OF THICK \otimes -IDEALS

If \mathcal{T} is a \otimes -triangulated category and $S \subseteq \mathcal{T}$ is a subset, then define $\langle S \rangle_\otimes \subseteq \mathcal{T}$ to be the smallest thick \otimes -ideal of \mathcal{T} containing S .

In order to prove Theorem 1.1, we require the following lemma, which is analogous to [Tho97, Lem. 3.14].

Lemma 3.1. *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. If $P, Q \in \mathrm{D}_{\mathrm{qc}}(X)^c$ and $\mathrm{supp}(P) \subseteq \mathrm{supp}(Q)$, then $\langle P \rangle_\otimes \subseteq \langle Q \rangle_\otimes$.*

Proof. Argue exactly as in [Tho97, Lem. 3.14] (cf. [Nee92a, Lem. 1.2]), but this time using Theorem 2.3 instead of [Tho97, Thm. 3.8]. \square

The following example shows Lemma 3.1 cannot be extended to $P, Q \in \mathrm{Perf}(X)$ when X is non-tame. It also shows that Thomason's Classification (Theorem 1.1) does not hold for $\mathrm{Perf}(X)$ in this case too.

Example 3.2. Let $x: \operatorname{Spec} \mathbb{F}_2 \rightarrow X$ be as in Example 2.4. Let $P = \mathcal{O}_X$ and let $Q = x_* \mathcal{O}_{\operatorname{Spec} \mathbb{F}_2}$. Then $P, Q \in \operatorname{Perf}(X)$ and $\operatorname{supph}(P) = \operatorname{supph}(Q)$. Note that $Q \in D_{\operatorname{qc}}(X)^c$ and $P \notin D_{\operatorname{qc}}(X)^c$. Since $D_{\operatorname{qc}}(X)^c$ is a thick \otimes -ideal of $\operatorname{Perf}(X)$, it follows that $\langle Q \rangle_{\otimes} \subseteq D_{\operatorname{qc}}(X)^c$. But if $\langle P \rangle_{\otimes} = \langle Q \rangle_{\otimes}$, then $P \in D_{\operatorname{qc}}(X)^c$. But $P \notin D_{\operatorname{qc}}(X)^c$, thus we have a contradiction.

Following Thomason [Tho97, Thm. 3.15] (or Neeman [Nee92a, Thm. 1.5]), given Lemma 3.1, we can prove Theorem 1.1.

Proof of Theorem 1.1. If $Y \subseteq |X|$ is a Thomason subset, then define

$$\mathcal{J}_Y = \{P \in D_{\operatorname{qc}}(X)^c : \operatorname{supph}(P) \subseteq Y\}.$$

Clearly, \mathcal{J}_Y is a thick \otimes -ideal of $D_{\operatorname{qc}}(X)^c$. If \mathcal{T} is a thick \otimes -ideal of $D_{\operatorname{qc}}(X)^c$, then define

$$\varphi(\mathcal{T}) = \cup_{Q \in \mathcal{T}} \operatorname{supph}(Q).$$

By [HR14c, Lem. 3.5(3)], $\varphi(\mathcal{T})$ is a Thomason subset of $|X|$. It suffices to prove that $\mathcal{J}_{\varphi(\mathcal{T})} = \mathcal{T}$ and $\varphi(\mathcal{J}_Y) = Y$.

Obviously, $\mathcal{T} \subseteq \mathcal{J}_{\varphi(\mathcal{T})}$. For the reverse inclusion, if $P \in \mathcal{J}_{\varphi(\mathcal{T})}$, then $\operatorname{supph}(P) \subseteq \cup_{Q \in \mathcal{T}} \operatorname{supph}(Q)$. Since $\operatorname{supph}(P)$ and $\operatorname{supph}(Q)$ are constructible for every $Q \in \mathcal{T}$, it follows that there is a finite subset $J \subseteq \mathcal{T}$ such that

$$\operatorname{supph}(P) \subseteq \cup_{Q \in J} \operatorname{supph}(Q) = \operatorname{supph}(\oplus_{Q \in J} Q).$$

By Lemma 3.1, $\langle P \rangle_{\otimes} \subseteq \langle \oplus_{Q \in J} Q \rangle_{\otimes} \subseteq \mathcal{T}$. Thus $P \in \mathcal{T}$ and $\mathcal{J}_{\varphi(\mathcal{T})} = \mathcal{T}$.

Obviously, $Y \supseteq \varphi(\mathcal{J}_Y)$. Since Y is Thomason, it is expressible as a union $\cup_{\alpha} Y_{\alpha}$ such that $|X| \setminus Y_{\alpha}$ is quasi-compact and open. By [HR14c, Thm. A], for every α there is a compact complex Q_{α} with support Y_{α} . It follows that if $y \in Y$, then $y \in \operatorname{supph}(Q_{\alpha}) \subseteq Y$ for some α . That is, $y \in \varphi(\mathcal{J}_Y)$ so $Y = \varphi(\mathcal{J}_Y)$. \square

4. THE BALMER SPECTRUM OF A TAME STACK

We will prove Theorem 1.2 using [BKS07, Prop. 6.1].

Proof of Theorem 1.2. Let $s: (|X|, \operatorname{supph}) \rightarrow (|\operatorname{Sp}^{\operatorname{Bal}}(\operatorname{Perf}(X))|, \sigma_X)$ be the uniquely induced morphism of support data, where σ_X denotes the universal support datum. By Theorem 1.1, $(|X|, \operatorname{supph})$ is classifying and by [LMB, Cor. 5.6.1 & 5.7.2] we know that $|X|$ is spectral. By [BKS07, Prop. 6.1], s is a homeomorphism. By definition, $\mathcal{O}_{\operatorname{Sp}^{\operatorname{Bal}}(\operatorname{Perf}(X))}$ is the sheafification of the presheaf

$$(i: U \subseteq X) \mapsto \operatorname{End}_{\operatorname{Perf}(X)/\ker(i^*) \cap \operatorname{Perf}(X)}(i^* \mathcal{O}_X).$$

Since $|X|$ has a basis consisting of quasi-compact open subsets, it is sufficient to identify $\operatorname{End}_{\operatorname{Perf}(X)/\ker(i^*) \cap \operatorname{Perf}(X)}(i^* \mathcal{O}_X)$ when i is a quasi-compact open immersion. By [HR14c, Lem. 6.7(2)], $\ker(i^*)$ is the localizing envelope of a set of objects with compact image in $D_{\operatorname{qc}}(X)$. By Thomason's Localization Theorem (e.g., [HR14c, Thm. 4.14] or [Nee92b, Thm. 2.1]), $\operatorname{Perf}(U)$ is the thick closure of $\operatorname{Perf}(X)/\ker(i^*) \cap \operatorname{Perf}(X)$. Since there are natural isomorphisms

$$\operatorname{End}_{\operatorname{Perf}(X)/\ker(i^*) \cap \operatorname{Perf}(X)}(i^* \mathcal{O}_X) \cong \operatorname{End}_{\operatorname{Perf}(U)}(\mathcal{O}_U) \cong \operatorname{End}_{\mathcal{O}_U}(\mathcal{O}_U) = \Gamma(U, \mathcal{O}_X),$$

the result follows. \square

Proof of Theorem 1.3. Since X has finite inertia, it has separated diagonal. By [Ryd13, Thm. 6.12], π is a separated universal homeomorphism, so X_{cs} is a quasi-compact and quasi-separated algebraic space. By [Ryd13, Thm. 6.12], the natural map $(|X|, \mathcal{O}_{X_{\operatorname{Zar}}}) \rightarrow (|X_{\operatorname{cs}}|, \mathcal{O}_{(X_{\operatorname{cs}})_{\operatorname{Zar}}})$ is an isomorphism of locally ringed spaces. By Theorem 1.2, the result follows. \square

APPENDIX A. TAME STACKS AND COARSE SPACES

In this appendix, we establish some basic results about $R(\pi_{qc})_*$, where $\pi: X \rightarrow X_{cs}$ is the coarse space of a quasi-separated algebraic stack X with finite inertia. Our first result, however, is a useful lemma that characterises the compact objects on a certain class of algebraic stacks, which includes BG for all finite groups G . This is likely known, though we are unaware of a reference for this result in the generality required.

Lemma A.1. *Let W be an algebraic stack and let $v: V \rightarrow W$ be a finite and faithfully flat morphism of finite presentation, where V is an affine scheme. If $M \in D_{qc}(W)^c$, then M is quasi-isomorphic to a direct summand of $R(v_{qc})_*P$ for some $P \in \text{Perf}(V)$.*

Proof. If $P \in \text{Perf}(V)$, then $R(v_{qc})_*P \in D_{qc}(W)^c$ [HR14c, Prop. 3.3 & Ex. 4.8]. Thus, let $\mathcal{T} \subseteq D_{qc}(W)^c$ be the subcategory with objects those $N \in D_{qc}(W)^c$ that are quasi-isomorphic to direct summands of $R(v_{qc})_*P$ for some $P \in \text{Perf}(V)$. Clearly, \mathcal{T} is closed under shifts and direct summands. We now prove that \mathcal{T} is triangulated. Thus let $f: N' \rightarrow N$ be a morphism in \mathcal{T} and complete it to a distinguished triangle

$$N' \xrightarrow{f} N \xrightarrow{c} N'' \xrightarrow{\partial} N'[1].$$

We now prove that $N'' \in \mathcal{T}$. By assumption, there are $P, P' \in \text{Perf}(V)$ and $C, C' \in D_{qc}(W)^c$ and quasi-isomorphisms $N \oplus C \simeq R(v_{qc})_*P$, $N' \oplus C' \simeq R(v_{qc})_*P'$. It follows that there is a distinguished triangle

$$N' \oplus C' \xrightarrow{f \oplus 0} N \oplus C \xrightarrow{c \oplus \text{id}_C \oplus 0} N'' \oplus C \oplus C'[1] \xrightarrow{\partial \oplus p_{C'[1]}} N' \oplus C'[1],$$

where $p_{C'[1]}: C \oplus C'[1] \rightarrow C'[1]$ is the natural projection. In particular, we are reduced to the situation where $N' = R(v_{qc})_*P'$ and $N = R(v_{qc})_*P$. In this case, the morphism $f: N' \rightarrow N$ by duality induces a morphism $\tilde{f}: P' \rightarrow v^*R(v_{qc})_*P$. It follows that the composition $R(v_{qc})_*P' \xrightarrow{\tilde{f}} R(v_{qc})_*P \rightarrow R(v_{qc})_*v^*R(v_{qc})_*P$ is the map $R(v_{qc})_*\tilde{f}$. Now form a distinguished triangle

$$P' \xrightarrow{\tilde{f}} v^*R(v_{qc})_*P \xrightarrow{k} K \xrightarrow{\delta} P'[1].$$

Since the morphism $R(v_{qc})_*P \rightarrow R(v_{qc})_*v^*R(v_{qc})_*P$ admits a retraction, there exists a $Q \in D_{qc}(W)^c$ and a quasi-isomorphism $R(v_{qc})_*v^*R(v_{qc})_*P \simeq R(v_{qc})_*P \oplus Q$. There is an induced morphism of distinguished triangles

$$\begin{array}{ccccccc} R(v_{qc})_*P' & \xrightarrow{R(v_{qc})_*\tilde{f}} & R(v_{qc})_*v^*R(v_{qc})_*P & \xrightarrow{R(v_{qc})_*k} & R(v_{qc})_*K & \xrightarrow{R(v_{qc})_*\delta} & R(v_{qc})_*P'[1] \\ \parallel & & \sim \downarrow & & \downarrow & & \downarrow \\ R(v_{qc})_*P' & \xrightarrow{f \oplus 0} & R(v_{qc})_*P \oplus Q & \xrightarrow{c \oplus \text{id}_Q} & N'' \oplus Q & \xrightarrow{\partial + 0} & R(v_{qc})_*P'[1]. \end{array}$$

It follows that $R(v_{qc})_*K \simeq N'' \oplus Q$ and so $N'' \in \mathcal{T}$. By [HR14c, Ex. 6.5 & Prop. 6.6], $D_{qc}(W)$ is compactly generated by $v_*\mathcal{O}_V$. But Thomason's Theorem [Nee92b, Thm. 2.1] implies that $D_{qc}(W)^c$ is the smallest thick subcategory containing $v_*\mathcal{O}_V$. The result follows. \square

The following result was suggested to us by David Rydh.

Theorem A.2. *If X be a quasi-separated algebraic stack with finite inertia and coarse space $\pi: X \rightarrow X_{cs}$, then the restriction of $R(\pi_{qc})_*$ to $D_{qc}(X)^c$ is t -exact.*

Proof. By [HR14c, Lem. 1.2(4)], this may be checked étale-locally on X_{cs} . Thus, we may assume that X_{cs} is an affine scheme. Since π is a universal homeomorphism, it follows that X is quasi-compact. Also, since X has finite inertia, it has quasi-finite and separated diagonal. By Theorem B.5, there exist morphisms of algebraic stacks $V \xrightarrow{v} W \xrightarrow{p} X$, such that V is an affine scheme, v is finite faithfully flat and finitely represented and p is a representable, separated and finitely presented Nisnevich covering. By [Ryd13, Prop. 6.5], we may further assume that p is fixed-point reflecting. We now apply [Ryd13, Thm. 6.10] to conclude that the following diagram

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ \omega \downarrow & & \downarrow \pi \\ W_{\text{cs}} & \xrightarrow{p_{\text{cs}}} & X_{\text{cs}} \end{array}$$

is cartesian and p_{cs} is representable, separated, étale and of finite presentation. Thus, it suffices to prove the result on W .

Let $M \in D_{\text{qc}}(W)^c \cap D_{\text{qc}}^{\leq 0}(W)$. By Lemma A.1, we may assume that there is map $i: M \rightarrow R(v_{\text{qc}})_*P$, where $P \in \text{Perf}(V)$, that admits a retraction r . It follows that the composition $M \xrightarrow{i} R(v_{\text{qc}})_*P \rightarrow \tau^{>0}R(v_{\text{qc}})_*P$ is the zero map. Thus the induced map $R(\omega_{\text{qc}})_*M \rightarrow R(\omega_{\text{qc}})_*\tau^{>0}R(v_{\text{qc}})_*P$ is the 0 map. But v and $\omega \circ v$ are affine, so there is a natural quasi-isomorphism $\tau^{>0}R(\omega_{\text{qc}})_*R(v_{\text{qc}})_*P \simeq R(\omega_{\text{qc}})_*\tau^{>0}R(v_{\text{qc}})_*P$. The resulting map $\tau^{>0}R(\omega_{\text{qc}})_*M \rightarrow \tau^{>0}R(\omega_{\text{qc}})_*R(v_{\text{qc}})_*P$ is 0 and also coincides with $\tau^{>0}R(\omega_{\text{qc}})_*(i)$, which admits a retraction $\tau^{>0}R(\omega_{\text{qc}})_*(r)$. In particular, $\tau^{>0}R(\omega_{\text{qc}})_*M \simeq 0$ and the result follows. \square

In [AOV08], they work with a more restrictive definition of tame, rendering the following corollary a tautology. Indeed, they assume that X has finite inertia and is locally of finite presentation over a base scheme S and that $\pi: X \rightarrow X_{\text{cs}}$ is such that π_* is exact on quasi-coherent sheaves. In our case, we make none of these assumptions, thus it is non-trivial.

Corollary A.3. *Let X be a quasi-separated algebraic stack with finite inertia and coarse space $\pi: X \rightarrow X_{\text{cs}}$. The following are equivalent:*

- (1) X is tame,
- (2) $\pi_*: \text{QCoh}(X) \rightarrow \text{QCoh}(X_{\text{cs}})$ is exact,
- (3) $R\pi_*: D_{\text{qc}}^+(X) \rightarrow D_{\text{qc}}^+(X_{\text{cs}})$ is t -exact, and
- (4) $R(\pi_{\text{qc}})_*: D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(X_{\text{cs}})$ is t -exact.

Proof. We begin with some preliminary reductions. The morphism π is a separated universal homeomorphism [Ryd13, Thm. 6.12], so X_{cs} is a quasi-separated algebraic space and π is quasi-compact and quasi-separated. Thus by [HR14c, Lem. 1.2(2)], (3) \Rightarrow (4) and by [HR14c, Thm. 2.6(2)] we have (4) \Rightarrow (3). Clearly, (1) may be verified after passing to an affine étale presentation of X_{cs} , and similarly for (2) and (3) [HR14c, Lem. 1.2(4) & Lem. 2.2(6)]. We may consequently assume that X_{cs} is an affine scheme. Since π has finite diagonal, it has affine diagonal, so we have (2) \Leftrightarrow (3) [HNR14, Prop. 2.1]. By [HR14b, Thm. C(1) \Rightarrow (3)], we now obtain (2) \Rightarrow (1). It remains to address (1) \Rightarrow (2).

Arguing exactly as in the proof of Theorem A.2, we may further assume that X admits a finite, faithfully flat and finitely presented cover $v: V \rightarrow X$, where V is an affine scheme. Since X is tame, $\mathcal{O}_X \in D_{\text{qc}}(X)^c$. By Theorem A.2, it follows that the induced morphism $\mathcal{O}_X \rightarrow v_*\mathcal{O}_V$ admits a retraction. If $M \in \text{QCoh}(X)$, then it follows immediately that the natural map $M \rightarrow v_*v^*M$ admits a retraction. Thus, if $f: M \rightarrow N$ is a surjection in $\text{QCoh}(X)$, then f is a retraction of the surjection

v_*v^*f . Since $\pi \circ v$ is affine, $\pi_*v_*v^*f$ is surjective. In particular, π_*f is a retraction of a surjection, thus is surjective. The result follows. \square

APPENDIX B. THE INDUCTION PRINCIPLE

The *induction principle* [Stacks, Tag 08GL] for algebraic spaces is closely related to the étale dévissage results of [Ryd11a]. When working with derived categories, where locality results are often quite subtle, it is often advantageous to have the strongest possible criteria at your disposal. In this appendix, we will prove the following induction principle for stacks with quasi-finite and separated diagonal. In [HR14a], this will be generalized to stacks with non-separated diagonals and put into a broader context.

Before state this result, we require some notation. Fix an algebraic stack S . If P_1, \dots, P_r is a list of properties of morphisms of algebraic stacks to S , let $\mathbf{Stack}_{P_1, \dots, P_r/S}$ denote the full 2-subcategory of the category of algebraic stacks over S whose objects are those $(x: X \rightarrow S)$ such that x has properties P_1, \dots, P_r . The following abbreviations will be used: ét (étale), qff (quasi-finite flat), sep (separated), fp (finitely presented) and rep (representable).

Theorem B.1 (Induction principle). *Let S be a quasi-compact algebraic stack with quasi-compact and separated diagonal. If S has quasi-finite diagonal, let $\mathbf{E} = \mathbf{Stack}_{\text{rep, sep, qff, fp}/S}$; or if S is Deligne–Mumford, let $\mathbf{E} = \mathbf{Stack}_{\text{rep, sep, ét, fp}/S}$. Let $\mathbf{D} \subseteq \mathbf{E}$ be a full subcategory satisfying the following properties:*

- (I1) *if $(X' \rightarrow X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$;*
- (I2) *if $(X' \rightarrow X) \in \mathbf{E}$ is finite and surjective, where X' is an affine scheme, then $X \in \mathbf{D}$; and*
- (I3) *if $(U \xrightarrow{i} X), (X' \xrightarrow{f} X) \in \mathbf{E}$, where i is an open immersion and f is étale and an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U, X' \in \mathbf{D}$.*

Then $\mathbf{D} = \mathbf{E}$. In particular, $S \in \mathbf{D}$.

Proof. Combine Lemma B.3 with Theorem B.5. \square

We wish to point out that Theorem B.1 relies on the existence of coarse spaces for stacks with finite inertia (i.e., the Keel–Mori Theorem [KM97, Ryd13]).

B.1. Nisnevich coverings. It will be useful to consider some variants and refinements of [KØ12, §§7–8].

If $p: W \rightarrow X$ is a representable morphism of algebraic stacks, then a *splitting sequence* for p is a sequence of quasi-compact open immersions

$$\emptyset = X_0 \subseteq X_1 \subseteq \dots \subseteq X_r = X$$

such that p restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a section for each $i = 1, \dots, r$. In this situation, we say that p has a splitting sequence of length r . An étale and representable morphism of algebraic stacks $p: W \rightarrow X$ is a *Nisnevich covering* if it admits a splitting sequence.

Example B.2. Let X be a quasi-compact and quasi-separated scheme. Then there exists an affine scheme W and a Nisnevich covering $p: W \rightarrow X$. Indeed, taking $W = \coprod_{i=1}^n U_i$, where the $\{U_i\}$ form a finite affine open covering of X gives the claim.

The following lemma is proved by a straightforward induction on the length of the splitting sequence.

Lemma B.3 (Nisnevich dévissage). *Let S be a quasi-compact and quasi-separated algebraic stack. Let \mathbf{E} be $\mathbf{Stack}_{\text{rep, ét, fp}/S}$ or $\mathbf{Stack}_{\text{rep, sep, ét, fp}/S}$. Let $\mathbf{D} \subseteq \mathbf{E}$ be a full 2-subcategory with the following properties:*

- (N1) if $(X' \rightarrow X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$; and
 (N2) if $(U \xrightarrow{i} X), (X' \xrightarrow{f} X) \in \mathbf{E}$, where i is an open immersion and f is an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U, X' \in \mathbf{D}$.

If $p: W \rightarrow X$ is a Nisnevich covering in \mathbf{E} and $W \in \mathbf{D}$, then $X \in \mathbf{D}$.

The following lemma will also be useful.

Lemma B.4. *Let $p: W \rightarrow X$ be a Nisnevich covering of algebraic stacks.*

- (1) *If $f: X' \rightarrow X$ is a morphism of algebraic stacks, then the pull back $p': W' \rightarrow X'$ of p along f is a Nisnevich covering.*
- (2) *Let $w: W' \rightarrow W$ be a Nisnevich covering of finite presentation. If p is of finite presentation and X is quasi-compact and quasi-separated, then $p \circ w: W' \rightarrow X$ is a Nisnevich covering.*

B.2. Presentations. The following theorem refines [Ryd11a, Thm. 7.2] and will be crucial for the proof of Theorem B.1.

Theorem B.5. *Let X be a quasi-compact algebraic stack with quasi-finite and separated diagonal. Then there exist morphisms of algebraic stacks*

$$V \xrightarrow{v} W \xrightarrow{p} X$$

such that

- V is an affine scheme;
- v is finite, flat, surjective and of finite presentation; and
- p is a separated Nisnevich covering of finite presentation.

In addition, if S is a Deligne–Mumford stack, it can be arranged that v is also étale.

Proof. The proof is similar to [Ryd13, Prop. 6.11] and [Ryd11a, Thm. 7.3].

By [Ryd11a, Thm. 7.1], there is an affine scheme U and a representable, separated, quasi-finite, flat, surjective and morphism $u: U \rightarrow X$ of finite presentation. Let $W = \underline{\text{Hilb}}_{U/X}^{\text{open}} \rightarrow X$ be the subfunctor of the relative Hilbert scheme parameterizing open and closed immersions to U over X . It follows that $p: W \rightarrow X$ is étale, representable and separated [Ryd11b, Cor. 6.2].

We now prove that p is a Nisnevich covering. To see this, we note that there exists a sequence of quasi-compact open immersions

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that the restriction of u to $Z_i = (X_i \setminus X_{i-1})_{\text{red}}$ for $i = 1, \dots, r$ is finite, flat and finitely presented. By definition of $p: W \rightarrow X$, it follows immediately that $p|_{Z_i}$ admits a section corresponding to $u|_{Z_i}$ and so p is a separated Nisnevich covering.

Let $v: V \rightarrow W$ be the universal family, which is finite, flat, surjective and of finite presentation. Also, $V \rightarrow U$ is representable, étale and separated [Ryd11b, Cor. 6.2]. Suitably shrinking W , we obtain a separated Nisnevich covering $p: W \rightarrow X$ of finite presentation fitting into a 2-commutative diagram

$$(B.1) \quad \begin{array}{ccc} V & \xrightarrow{q} & U \\ v \downarrow & & \downarrow u \\ W & \xrightarrow{p} & X, \end{array}$$

and q is étale, separated and surjective. By Zariski’s Main Theorem [LMB, Thm. A.2], q is quasi-affine. By [Ryd13, Thm. 5.3], W has a coarse space $\pi: W \rightarrow W_{\text{cs}}$ such that W_{cs} is a quasi-affine scheme and $\pi \circ v$ is affine. By Example B.2 and Lemma B.4, we may further reduce to the situation where W_{cs} is an affine scheme. Since $\pi \circ v$ is affine, the result follows. \square

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